Complex Analysis

Preliminary Examimation

May 2005

Do all problems. Present adequate work to justify your answers.

Notation: $B(0;r) = \{z \in \mathbb{C} : \ |z| < r\}$, $U = \{z \in \mathbb{C} : \ \text{Im } z > 0\}$,
$\mathbb{D} = B(0;1)$, $\text{ann}(a;\alpha,\beta) = \{z \in \mathbb{C} : \ \alpha < |z-a| < \beta\}$.

1. Evaluate the integral $\displaystyle\int_0^\infty \frac{\sin x}{x(1+x^2)} dx$.
2. Give an example of each of the following, or state why an example does not exist. (These are short answer questions.)
   a) A function with a pole of order two at each integer.
   b) An entire function which maps the open ball $\mathbb{D}$ onto $\{z : \ \text{Re } z \geq 0\}$.
   c) An analytic, one-to-one function which maps $\mathbb{D}$ onto $\mathbb{C}$.
   d) A function with an essential singularity at $z = 0$.
   e) A function which is analytic on $\mathbb{C}$, but which is not meromorphic on the Riemann sphere.
3. Find the Laurent expansion of $f(z) = \dfrac{1}{z^2(1-z)}$ in the given annulus.
   a) $\text{ann}(0;0,1)$.
   b) $\text{ann}(0;1,\infty)$.
4. Let $S = \{z = re^{i\theta} : \ 0 < r < 3, \ |\ \theta| < \pi/6\} \setminus (0,1]$. Find a one-to-one, conformal map f which takes S onto U such that $f(2) = i$.
5. Prove that $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, \dots$.
6. Let $\displaystyle\sum_{n=0}^{\infty} a_n(z-1)^n$ be the series expansion for $\dfrac{1}{\cos z}$ about $z = 1$.
   a) What is the radius of convergence of the series?
   b) Does the series $\displaystyle\sum_{n=0}^{\infty} |\ a_n|$ converge?
7. Show that every meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.
8. How many roots does the equation $e^z = (e+1)z^5$ have in $\mathbb{D}$?
9. Suppose that f is an analytic map from a region Ω into a region G and suppose that u is harmonic on G. Show that $u \circ f$ is harmonic on Ω.
10. Suppose that f is an entire function and that $|f(z)+1| > 1$ for all $z \in \mathbb{C}$. Show that f is constant.