Preliminary Examination 2001
Complex Analysis

Do all problems.

Notation.

$\mathbb{C} = \{z : z \text{ is a complex number}\}$ $\qquad\qquad$ $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

1. Prove that if f is entire and $f(-z) = f(z)$ for all z, then there is an entire function g so that $f(z) = g(z^2)$ for all z.

2. Let $D = \mathbb{D} \cap \{z : \text{ Im } z > 0\}$. Find the image of D under the map $f(z) = \exp(\dfrac{i - iz}{z+1})$.

3. Let $D = \{z : 0 < \arg \text{z} < 3\pi/2\}$. Find a function u which is continuous on $\bar{D} \setminus \{0\}$, harmonic on D, and satisfies $u(x, 0) = 1$ for $x > 0$ and $u(0, y) = 0$ for $y < 0$, where $z = x + iy$.

4. Let f be an analytic, one-to-one mapping from $\mathbb{D}$ onto a simply connected region G such that $f(0) = 0$. Let $d = \text{dist}(0, \mathbb{C} \setminus G)$. Show that $|f'(0)| \geq d$.

5. Suppose that f is an entire function and Im $f(z) \neq 0$ whenever $|z| \neq 1$. Prove that f is constant.

6. Suppose f is analytic in $\mathbb{C} \setminus \{0\}$ and satisfies
$$|f(z)| \leq |z|^2 + \frac{1}{|z|^2}$$
for $z \neq 0$. If f is an odd function, what form must the Laurent series of f have?

7. Evaluate the following integral, justifying all of your steps.
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + x + 1} dx$$

8. Suppose $\{f_n\}$ is a sequence of analytic functions on a region D such that there exists a positive constant M with the property that
$$\int\int_D |f_n(z)|^2 dxdy \leq M \text{ for all n.}$$
Show that $\{f_n\}$ has a subsequence that converges uniformly on compact subsets of D.

   Hint: If f is analytic in a neighborhood of a closed ball $\overline{B(a; R)}$, show that
$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r dr d\theta.$$

9. Show that there is no one-to-one analytic function which maps $A = \{z : 0 < |z| < 1\}$ onto $B = \{z : 1 < |z| < 2\}$.

10. Suppose f is analytic on $|z| < 1$ and continuous on $|z| \leq 1$. Assume $f(z) = 0$ on an open arc on the circle $|z| = 1$. Prove that $f \equiv 0$.