Preliminary Examination 2000
Complex Analysis

Do all problems.

Notation.
$\mathbb{C} = \{z : z \text{ is a complex number}\}$ $\qquad$ $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

1. Show that if u is a real-valued harmonic function in a domain $\Omega \subset \mathbb{C}$ such that u^2 is harmonic in Ω, then u is constant.

2. For $|z| < 1$ let $f(z) = \frac{1}{1-z} \exp\left[-\frac{1}{1-z}\right]$, and for $0 \le \theta < 2\pi$ let $\ell_\theta = \{z : z = re^{i\theta}, 0 \le r < 1\}$. Show that f is bounded on each set ℓ_θ. Is f bounded on $\mathbb{D}$? Explain.

3. Let $\Omega \subset \mathbb{C}$ be the intersection of the two disks of radius 2 whose centers are at $z = 1$ and $z = -1$. Find an explicit conformal mapping of Ω onto the upper-half plane.

4. Does there exist a function f that is analytic in a neighborhood of $z = 0$, for which
   (a) $f(1/n) = f(-1/n) = 1/n^2$ for all sufficiently large integers n?
   (b) $f(1/n) = f(-1/n) = 1/n^3$ for all sufficiently large integers n?
   In each case, either give an example or prove that no such function exists.

5. (a) Let f be analytic on $\mathbb{D}$ with $\lim_{|z|\to 1^-} f(z) = 0$. Prove $f \equiv 0$.
   (b) Let g be analytic on $\mathbb{D}$. Prove that the statement $\lim_{|z|\to 1^-} g(z) = \infty$ is impossible.

6. Let $f : \mathbb{D} \to \mathbb{D}$ be analytic. Suppose there exists $z_0 \in \mathbb{D}$ with $f(z_0) = z_0$ and $f'(z_0) = 1$. Prove that $f(z) \equiv z$.

7. Find all Laurent expansions of $\frac{1}{(z-2)(z-3)}$ in powers of z and state where they converge.

8. Use the Theorem of Residues and an appropriate contour to evaluate
$$\int_{-\infty}^{\infty} \frac{\sqrt{x+i}}{1+x^2} dx \ ,$$
   where on $\{\text{Im } z > 0\}$, we choose the branch of $\sqrt{z+i}$ whose value at 0 is $e^{\pi i/4}$. Describe your method carefully, and include verification of all relevant limit statements.

9. Show that there exists an unbounded analytic function f on $\mathbb{D}$ such that
$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty,$$
   where dA is area measure on $\mathbb{D}$.

10. Show that every function that is meromorphic on the extended complex plane is rational.