Preliminary Examination 1999
Complex Analysis

Do all problems.

Notation:

$\mathbb{R} = \{x : x \text{ is a real number}\}$ $\qquad$ $\mathbb{C} = \{z : z \text{ is a complex number}\}$

$B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ $\qquad$ $\text{ann}(a, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z-a| < r_2\}$

$D = B(0,1)$

For $G \subset \mathbb{C}$, let $\mathscr{A}(G)$ denote the set of analytic functions on G (mapping G to $\mathbb{C}$) and let $\mathscr{Har}(G)$ denote the set of harmonic functions on G (mapping G to $\mathbb{R}$).

1. Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ converges on $|z| < R$ where z and the a_n are complex numbers. If $b_n \in \mathbb{C}$ is such that $|b_n| < n^2 |a_n|$ for all n, prove that $\sum_{n=0}^{\infty} b_n z^n$ converges for $|z| < R$.

2. Let $f, g \in \mathscr{A}(D)$. For $0 < r < 1$, let $\Gamma_r = \partial B(0,r)$.

   a) Prove that the integral $\frac{1}{2\pi} \int_{\Gamma_r} \frac{1}{w} f(w) g(\frac{z}{w})\, dw$ is independent of r provided that $|z| < r < 1$ and that it defines an analytic function $h(z)$, $|z| < 1$.

   b) Prove or supply an counterexample: if $f \not\equiv 0$ and $g \not\equiv 0$, then $h \not\equiv 0$.

3. Let $f \in \mathscr{A}(D)$. Suppose that $|f(z)| \le \frac{1}{(1-|z|)^{1/2}}$. Prove that there exists a M (independent of f) such that $|f'(z)| \le \frac{M}{(1-|z|)^{3/2}}$.

4. Let $f \in \mathscr{A}(D)$ such that $f(D) \subset D$, $f(0) = 0$. Prove for $z \in D$ that $|f(z) + f(-z)| \le 2|z|^2$ with equality if and only if $f(z) = \lambda z^2$ for some λ with $|\lambda| = 1$.

5. In which quadrant do the roots of $p(z) = z^4 + 2z + 1$ lie?

6. Let $f \in \mathscr{A}(D)$ and suppose that f is not the identity map. How many fixed points can f have?

7. Let f be an entire function. Suppose that f has a root at $z = +i$ and $z = -i$. Let $M = \max_{|z|=2} |f(z)|$. Prove that $|f(z)| \leq \frac{M}{3} |z^2 + 1|$ on $B(0,2)$.

8. Prove that $\prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$ is an entire function and find its zeros, counting multiplicity.

9. Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + 1} dx$.

10. Let $u \in \mathscr{Har}(\mathbb{C})$. Show that u can be positive on all of $\mathbb{C}$ only if u is constant.