SIAM J. MATH. ANAL.
Vol. 31, No. 3, pp. 693–699

AN INEQUALITY INVOLVING THE GENERALIZED HYPERGEOMETRIC FUNCTION AND THE ARC LENGTH OF AN ELLIPSE*

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Abstract. In this paper we verify a conjecture of M. Vuorinen that the Muir approximation is a lower approximation to the arc length of an ellipse. Vuorinen conjectured that $f(x) = {}_2F_1(\frac{1}{2}, -\frac{1}{2}; 1; x) - [(1 + (1 - x)^{3/4})/2]^{2/3}$ is positive for $x \in (0, 1)$. The authors prove a much stronger result which says that the Maclaurin coefficients of f are nonnegative. As a key lemma, we show that ${}_3F_2(-n, a, b; 1 + a + b, 1 + \epsilon - n; 1) > 0$ when $0 < ab/(1 + a + b) < \epsilon < 1$ for all positive integers n.

Key words. hypergeometric, approximations, elliptical arc length

AMS subject classifications. 33C, 41A

PII. S0036141098341575

1. Introduction. Let a and b be the semiaxes of an ellipse with eccentricity $e = \sqrt{a^2 - b^2}/a$. Let $L(a, b)$ denote the arc length of the ellipse. Without loss of generality we can take one of the semiaxes, say a, to be 1. Legendre's complete elliptic integral of the second kind can be defined by

$$E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t}\, dt.$$

Elliptic integrals are so named because of their connection with $L(a, b)$. In turn, these are related to Gauss's hypergeometric functions, ${}_2F_1$, defined by

$${}_2F_1(a_1, a_2; b_1; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n n!} z^n$$

with the Appell (or Pochhammer) symbol $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1, a \neq 0$. We shall need the generalized hypergeometric function, ${}_pF_q$, defined by

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdot (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

(see [12, p. 73]). It was noted by Maclaurin in 1742 (see [2]) that

$$L(1, b) = 4E(e) = 2\pi {}_2F_1(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2).$$

There are various references, books, and articles, which discuss the relationships between elliptic integrals and hypergeometric functions (see [3], [7]) and their role in

*Received by the editors July 7, 1998; accepted for publication (in revised form) December 17, 1998; published electronically March 6, 2000.
http://www.siam.org/journals/sima/31-3/34157.html

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applications to physics (see [11], [9]) and in geometric function theory (see [10], [3]). From antiquity several more easily computable approximations to $L(a,b)$ have been suggested. The Almkvist–Berndt survey article [2] has an extensive discussion of these approximations. These approximations and their historical and recent connections to the approximations of π can be found in the Borweins' book [6]. An excellent source for all of the above ideas is the Anderson–Vamanamurthy–Vuorinen book *Conformal Invariants, Inequalities, and Quasiconformal Mappings* [3].

In 1883, it was proposed by Muir (see [2]) that $L(1,b)$ could be simply approximated by $2\pi[(1+b^{3/2})/2]^{2/3}$. A close numerical examination of the error in this approximation lead M. Vuorinen to pose Problem 5.6 in [13]. This was announced at several international conferences. Letting $x = 1-b^2$, he asked whether the Muir approximation

$$g(x) = \left(\frac{1+(1-x)^{3/4}}{2}\right)^{2/3}$$

is a lower approximation for the value given by the hypergeometric function

$$h(x) = {}_2F_1\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1; x\right),$$

that is, whether

$$h(x) - g(x) \geq 0 \text{ for all } x \in (0,1).$$

We shall prove the following much stronger result.

Theorem 1.1. *Let* $g(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $h(x) = \sum_{n=0}^{\infty} A_n x^n$. *Then,*

$$a_k \leq A_k \text{ for all } k = 0,1,2,\ldots,n,\ldots. \tag{1.1}$$

In particular, the function $f(x) \equiv [h(x)-g(x)]/x^4$ *is convex and increasing from* $(0,1]$ *onto* $(\alpha,\beta]$, *where* $\alpha = 2^{-14} = 0.000061\cdots$ *and* $\beta = (2/\pi) - 2^{-2/3} = 0.006659\cdots$.

Remarks. The ideas and techniques used to prove Lemma 2.1 and Theorem 1.1 will be used in [5] to determine surprising hierarchical relationships among the 13 historical approximations to $L(a,b)$ discussed in [2]. These approximations range over four centuries from Kepler's in 1642 to Almkvist's in 1985 and include two from Ramanujan.

2. Proof of main results. The proof of Theorem 1.1 requires the following lemma.

Lemma 2.1. *Suppose* $a,b>0$. *Then, for any* ϵ *satisfying* $\frac{ab}{1+a+b} < \epsilon < 1$,

$${}_3F_2(-n,a,b;1+a+b,1+\epsilon-n;1) > 0 \quad \textit{for all integers } n \geq 1.$$

For the reader's convenience, we include the following classical identities.

Identity 1 (see [1, p. 558, eq. (15.2.24)]). If $|z|<1$, then

$$(c-b-1)\cdot {}_2F_1(a,b;c;z) = (c-1)\cdot {}_2F_1(a,b;c-1;z) - b\cdot\ {}_2F_1(a,b+1;c;z).$$

Identity 2 (see [12, p. 60, Thm. 21]). If $|z|<1$, then

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\cdot {}_2F_1(c-a,c-b;c;z).$$

Identity 3 (see [8, p. 59, eq. (3.1.1)]). If $F = {}_3F_2$, then

$$F(-n, a, b; c, d; 1) = \frac{(d-b)_n}{(d)_n} F(-n, c-a, b; c, 1+b-d-n; 1).$$

Identity 4 (see [12, p. 82, eq. (14)]). If $F = {}_3F_2$ and $|z| < 1$, then

$$\begin{aligned}(a_1 - a_2) \cdot F(a_1, a_2, a_3; b_1, b_2; z) \\ = a_1 \cdot F(a_1+1, a_2, a_3; b_1, b_2; z) - a_2 \cdot F(a_1, a_2+1, a_3; b_1, b_2; z).\end{aligned}$$

Proof of Lemma 2.1. Using an idea suggested in [4], we let $F = {}_3F_2$ and consider the generating function

$$f(r) = \sum_{n=0}^{\infty} \frac{-(-\epsilon)_n}{n!} F(-n, a, b; 1+a+b, 1+\epsilon-n; 1) r^n = \sum_{n=0}^{\infty} c_n r^n,$$

where $|r| < 1$. Note that $-(-\epsilon)_n > 0$ for $0 < \epsilon < 1$ and for all $n \geq 1$. Thus we seek to verify that $c_n > 0$ for all $n \geq 1$.

In this direction, we have

$$\begin{aligned}
f(r) &= \sum_{n=0}^{\infty} \frac{-(-\epsilon)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{(a+b+1)_k (1+\epsilon-n)_k k!} r^n \\
&= \sum_{n=0}^{\infty} \frac{-(-\epsilon)_n}{(1)_n} \sum_{k=0}^{n} \frac{\frac{(-1)^k (1)_n}{(1)_{n-k}} (a)_k (b)_k}{(a+b+1)_k \frac{(-1)^k (-\epsilon)_n}{(-\epsilon)_{n-k}} k!} r^n \\
&\qquad \left\{\text{using } (\alpha)_{n-k} = \tfrac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \text{ and } (1)_n = n!\right\} \\
&= -\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(a)_k (b)_k}{(a+b+1)_k k!} r^k \right) \left(\frac{(-\epsilon)_{n-k}}{(n-k)!} r^{n-k} \right) \\
&= -\sum_{n=0}^{\infty} \left(\frac{(-\epsilon)_n}{(n)!} r^n \right) \sum_{k=0}^{\infty} \left(\frac{(a)_k (b)_k}{(a+b+1)_k k!} r^k \right) \qquad \text{(see [12, p. 57, eq. (2)])} \\
&= -(1-r)^{\epsilon} \; {}_2F_1(a, b; a+b+1; r).
\end{aligned}$$

Differentiating, we have

$$\begin{aligned}
f'(r) &= \epsilon (1-r)^{\epsilon-1} \; {}_2F_1(a, b; a+b+1; r) \\
&\quad - \frac{ab(1-r)^{\epsilon}}{(a+b+1)} \; {}_2F_1(a+1, b+1; a+b+2; r).
\end{aligned} \tag{2.1}$$

An application of Identity 1 followed by Identity 2 to ${}_2F_1(a+1, b+1; a+b+2; r)$ yields

$$\begin{aligned}
&\frac{ab(1-r)^{\epsilon}}{(a+b+1)} {}_2F_1(a+1, b+1; a+b+2; r) \\
&\quad = \frac{b(1-r)^{\epsilon}}{(a+b+1)} [(a+b+1) \cdot {}_2F_1(a+1, b+1; a+b+1; r) \\
&\quad - (b+1) \cdot {}_2F_1(a+1, b+2; a+b+2; r)] \\
&\quad = (1-r)^{\epsilon-1} \left[b \cdot {}_2F_1(a, b; a+b+1; r) - \frac{b(b+1)}{(a+b+1)} \cdot {}_2F_1(a, b+1; a+b+2; r) \right].
\end{aligned}$$

Thus (2.1) becomes

$$
\begin{aligned}
f'(r) &= (1-r)^{\epsilon-1}\Bigg[(\epsilon-b)\cdot {}_2F_1(a,b;a+b+1;r) \\
&\quad + \frac{b(b+1)}{(a+b+1)}\cdot {}_2F_1(a,b+1;a+b+2;r)\Bigg] \\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left[\frac{(b)_n(\epsilon-b)}{(a+b+1)_n}+\frac{b(b+1)(b+1)_n}{(a+b+1)(a+b+2)_n}\right]r^n \\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left[\frac{(b)_n(\epsilon-b)}{(a+b+1)_n}+\frac{(b+1)(b)_n(b+n)}{(a+b+1)_n(a+b+1+n)}\right]r^n \qquad (2.2)\\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+1)_n(a+b+1+n)n!} \\
&\quad \times\left[(\epsilon-b)(a+b+1+n)+(b+1)(b+n)\right]r^n \\
&= (1-r)^{\epsilon-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b+1)_n(a+b+1+n)n!} \qquad (2.3)\\
&\quad \times\left[\epsilon(a+b+1+n)+n-ab\right]r^n,
\end{aligned}
$$

where (2.2) makes use of $\alpha(\alpha+1)_n=(\alpha)_n(\alpha+n)$. If $\frac{ab}{a+b+1}<\epsilon<1$, then the expression in (2.3) is the product of two series with all positive Maclaurin series coefficients. Hence f' has all positive Maclaurin series coefficients which is equivalent to the desired result. ◻

Corollary 2.2. *Let $T_n = {}_3F_2\left(-n,\frac{3}{2},\frac{1}{2};2,\frac{5}{4}-n;1\right)$. Then, for all integers $n\geq 8$,*

$$T_{n+1}>T_n>0.$$

Proof. Let $F={}_3F_2$ and $B_n=\left(\frac{3}{4}-n\right)_n/\left(\frac{5}{4}-n\right)_n$. Using Identity 3, we have that

$$T_n=B_nF\left(-n,\tfrac{1}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right).$$

Direct calculation reveals that $T_9>T_8>0>T_7>\cdots>T_2=T_1$. Now suppose that $T_n>T_{n-1}>0$ for some $n\geq 9$ and note that $B_{n+1}/B_n=\left(n+\frac{1}{4}\right)/\left(n-\frac{1}{4}\right)$. Then,

$$
\begin{aligned}
T_{n+1} &= B_{n+1}F\left(-n-1,\tfrac{1}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right) \\
&= \frac{B_{n+1}}{\left(n+\frac{3}{2}\right)}\left[(n+1)F\left(-n,\tfrac{1}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)+\tfrac{1}{2}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right)\right] \qquad (2.4)\\
&= \frac{B_{n+1}(n+1)}{B_n\left(n+\frac{3}{2}\right)}T_n+\frac{B_{n+1}}{2\left(n+\frac{3}{2}\right)}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right) \\
&= \frac{\left(n+\frac{1}{4}\right)(n+1)}{\left(n-\frac{1}{4}\right)\left(n+\frac{3}{2}\right)}T_n+\frac{B_{n+1}}{2\left(n+\frac{3}{2}\right)}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right) \\
&> T_n+\frac{B_{n+1}}{2\left(n+\frac{3}{2}\right)}F\left(-n-1,\tfrac{3}{2},\tfrac{1}{2};2,\tfrac{1}{4};1\right),
\end{aligned}
$$

where (2.4) follows from Identity 4, and the inequality holds because $\frac{(n+1/4)(n+1)}{(n-1/4)(n+3/2)} > 1$ and $T_n > 0$. Since $B_{n+1} < 0$, we shall have that $T_{n+1} > T_n > 0$ provided we show that $F(-n-1, \frac{3}{2}, \frac{1}{2}; 2, \frac{1}{4}; 1) < 0$. To this end, we again apply Identity 3 to observe that

$$F\left(-n-1, \tfrac{3}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}; 1\right) = \frac{\left(-\frac{1}{4}\right)_{n+1}}{\left(\frac{1}{4}\right)_{n+1}} F\left(-n-1, \tfrac{1}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}-n; 1\right).$$

Since

$$\frac{\left(-\frac{1}{4}\right)_{n+1}}{\left(\frac{1}{4}\right)_{n+1}} < 0,$$

we need to show that

$$F\left(-n-1, \tfrac{1}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}-n; 1\right) > 0.$$

Letting $m = n+1$, $a = b = \frac{1}{2}$, and $\epsilon = \frac{1}{4}$, it follows from Lemma 2.1 that

$$F\left(-n-1, \tfrac{1}{2}, \tfrac{1}{2}; 2, \tfrac{1}{4}-n; 1\right) = F(-m, a, b; a+b+1, 1+\epsilon-m; 1) > 0.$$

Hence $T_{n+1} > T_n > 0$ for all integers $n \geq 8$ by induction. □

Proof of Theorem 1.1. Clearly,

$$A_n = \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n}{n!n!}.$$

Computing the logarithmic derivative of g we have

$$\frac{g'(x)}{g(x)} = -\frac{1}{2}\left(\frac{(1-x)^{-\frac{1}{4}}}{1+(1-x)^{\frac{3}{4}}}\right),$$

which implies

$$\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right)\left((1-x)^{\frac{1}{4}}+1-x\right) = -\frac{1}{2}\sum_{n=0}^{\infty} a_n x^n. \tag{2.5}$$

The coefficients of x^n of the left-hand side of (2.5) are obtained from the Cauchy product of the two terms. Solving for a_{n+1} yields (by extracting the nth and $(n-1)$st terms from the Cauchy product)

$$a_{n+1} = \frac{1}{2(n+1)}\left[\left(\frac{5}{4}n-\frac{1}{2}\right)a_n - \sum_{k=0}^{n-2}(k+1)a_{k+1}\frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}\right]. \tag{2.6}$$

We now verify (1.1) using an inductive argument. Clearly, the coefficients of the terms a_k in (2.6) are nonnegative. Computation gives: $a_0 = A_0 = 1$, $a_1 = A_1 = -1/4$, $a_2 = A_2 = -3/64$, $a_3 = A_3 = -5/2^8$, $a_4 = -11/2^{10}$ and $A_4 = -175/2^{14}$. Suppose that the inequality in (1.1) holds for $4 \leq k \leq n$. From (2.6) we have

(2.7) $a_{n+1} \leq$

$$\frac{1}{2(n+1)}\left[\left(\frac{5}{4}n-\frac{1}{2}\right)\frac{\left(\frac{1}{2}\right)_n\left(-\frac{1}{2}\right)_n}{n!n!} - \sum_{k=0}^{n-2}(k+1)\frac{\left(\frac{1}{2}\right)_{k+1}\left(-\frac{1}{2}\right)_{k+1}}{(k+1)!(k+1)!}\frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}\right].$$

We need to show that the right-hand side of (2.7) is less than or equal to $A_{n+1} = \frac{(\frac{1}{2})_{n+1}(-\frac{1}{2})_{n+1}}{(n+1)!(n+1)!}$, that is,

$$\left(\frac{5}{4}n - \frac{1}{2}\right) \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n}{n!n!} - 2(n+1) \frac{\left(\frac{1}{2}\right)_{n+1} \left(-\frac{1}{2}\right)_{n+1}}{(n+1)!(n+1)!} \leq \sum_{k=0}^{n-2} (k+1) \frac{\left(\frac{1}{2}\right)_{k+1} \left(-\frac{1}{2}\right)_{k+1}}{(k+1)!(k+1)!} \frac{\left(-\frac{1}{4}\right)_{n-k}}{(n-k)!}. \tag{2.8}$$

After adding the $(n-1)$st and nth terms of the right-hand side of (2.8) to inequality (2.8) and then simplifying, we use $(a)_{k+1} = (a+k)(a)_k$, $(a)_{n-k} = (-1)^k (a)_k / (1-a-n)_k$, the fact that $(-n)_k = 0$ for $k \geq n+1$, and the definition of ${}_3F_2$ to obtain

$$\frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n}{n!n!} \cdot \frac{(2n-1)}{4(n+1)} \leq -\left(\frac{1}{4}\right)\left(-\frac{1}{4}\right)_n \frac{{}_3F_2\left(-n, \frac{1}{2}, \frac{3}{2}; 2, \frac{5}{4} - n; 1\right)}{n!},$$

or equivalently

$${}_3F_2\left(-n, \tfrac{1}{2}, \tfrac{3}{2}; 2, \tfrac{5}{4} - n; 1\right) \geq \frac{\left(\frac{1}{2}\right)_n^2}{\left(-\frac{1}{4}\right)_n (n+1)!}. \tag{2.9}$$

Clearly, the right-hand side of (2.9) is negative for all $n \geq 1$. Inequality (2.9) can be explicitly verified for $0 \leq n \leq 7$. For $n \geq 8$, inequality (2.9) follows from Corollary 2.2. Thus, the inequality in (1.1) also holds for $k = n+1$. Hence, by induction (1.1) holds for all $k \in \mathbb{N} \cup \{0\}$.

Finally, the convexity and monotonicity of f are clear. By l'Hôpital's rule, $f(0^+) = A_4 - a_4 = 1/2^{14} = 1/16384$, while the value of $f(1)$ is clear. ∎

Acknowledgments. The authors would like to thank the referees for their many helpful suggestions which helped to considerably improve the paper.

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